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Unit - V

Complex Analysis - II

# TRANSFORMATION

## (2) Magnification (Definition) :-

The mapping $w = Az$ where $A$ is non-zero complex constant and $z \neq 0$ is said to be magnification writing $A$ and $z$ is exponential form as

$$A = ae^{i\alpha}, \quad z = re^{i\theta}$$

In 10 m, give proof

$$w = (ae^{i\alpha})(re^{i\theta})$$

$$w = ar\,e^{i(\alpha+\theta)}$$

* If $a > 1$, the mapping expands.
* If $a > 1$, the mapping contracts the region in z-plane
* If $\alpha > 0$, the rotation is in positive direction.

Thus the mapping preserves the shape of the region.

## (1) Translation : Definition :-

The mapping $w = z + B$ where $B$ is any complex constant is known as translation

writing, $w = u + iv$

$$z = x + iy$$

$$B = b_1 + ib_2$$

$$u + iv = (x + iy) + (b_1 + ib_2)$$

$$= (x + b_1) + i(y + b_2)$$

$$u = x + b_1 \quad v = y + b_2.$$

The image point is moved in the direction of $B$

Hence the image of any region in z plane is the same region in w-plane.

## Linear transformation :- Bilinear or Movious transformation :-

The transformation $w = Az + B$ $(A \neq 0)$ where A & B are complex constants is said to be linear transformation.

**Result :-**

Linear transformation is a combination of magnification and translation.

Eg :- Find the image of the rectangle, $x=0, x=1, y=0, y=2$ under the transformation $w = (1+i)z + 2$.

soln :-

$$w = (1+i)z + 2$$

Let $Z = (1+i)z$

$$w = Z + 2.$$

writing $(1+i)$, $z$ in exponential form.

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}.$$

$$1+i = \sqrt{2}e^{i\pi/4}$$

$$Z = (\sqrt{2}\, e^{i\pi/4})(re^{i\theta})$$

$$Z = \sqrt{2}\, r e^{i(\pi/4+\theta)}.$$

Thus this transformation expands the radius vector for a non-zero point $z$ by the factor $\sqrt{2}$. and rotates its counter clockwise $\frac{\pi}{4}$ radians about the origin.

The second transformation is a translation two ~~u~~nits to the right.

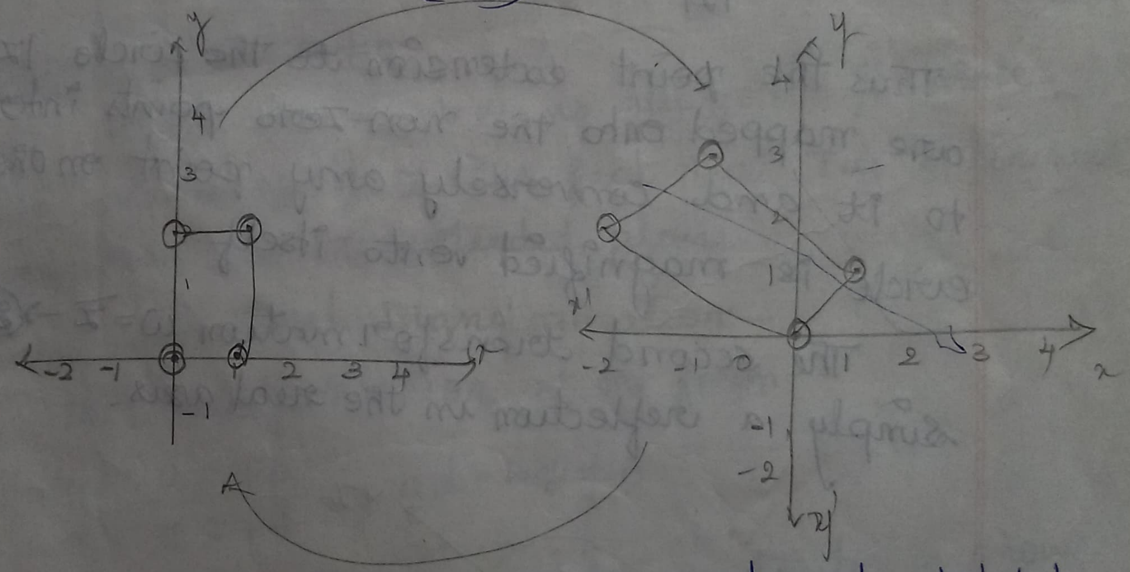
To find the image of rectangle under $Z = (1+i)z$

Image of $(0,0)$, $z = (1+i)(x+iy)$
$= 0 + i0$
$= (0,0)$.

Image of $(1,0)$, $z = (1+i)(1+0)$
$z = (1,1)$.

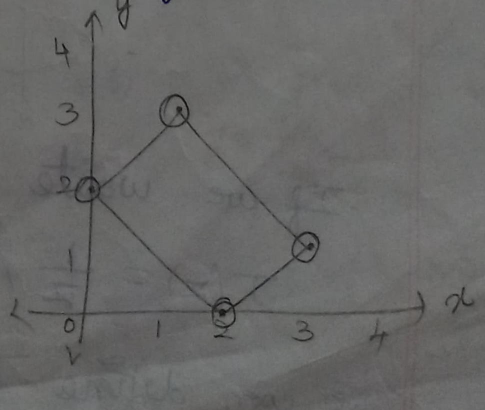
Image of $(1,2)$, $z = \cancel{(1+i)(i+2i)}$
$= (1+i)\, i\,(1+2)$
$= 1 - 2 + i(1+2)$
$= -1 + 3i$.
$\therefore\ z = (-1,3)$

Image of $(0,2)$, $z = (1+i)(0+2i)$
$= 2i - 2$
$= -2 + 2i$.
$z = (-2,2)$.




Now the image of the rectangle $A'B'C'D'$ under the mapping $w = z + 2$ is given by

$(0,0)$ goes to $(2,0)$
$(1,1)$ goes to $(3,1)$
$(-1,3)$ goes to $(1,3)$
$(-2,2)$ goes to $(0,2)$.



# Transformation $\omega = \frac{1}{z}$ or mapping $\omega = \frac{1}{z}$

**Soln** The equation $\omega = \frac{1}{z}$ establishes a one to one correspondence b/w non zero points on the $z$ and $\omega$ plane.

since $z\bar{z} = |z|^2$

The mapping can be described by the transformation

$$Z = \frac{z}{|z|^2} \to ①$$

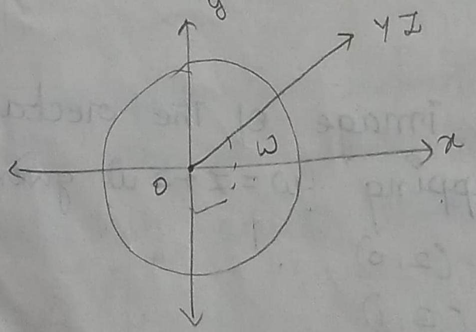
$$\omega = \bar{Z} \to ②$$

The first of these transformation is an inversion with respect to the limit circle $|z|=1$ i.e) the image of a non-zero point $z$ is the point $Z$ with the property.

$$|Z| = \frac{1}{|z|} \text{ and } \arg Z = \arg z.$$

Thus the point extension to the circle $|z|=1$ are mapped onto the non-zero points interior to it and conversely any point on the circle is magnified onto itself.

The second transformation $\omega = \bar{Z} \to ②$ is simply a reflection in the real axis.




If we write transformation ① as

$$T(z) = \frac{1}{z}, \quad z \neq 0 \to ③$$

we can define $T$ at the origin and that the point at infinitely. so as to be continuous on the extended complex plane

$$\lim_{z\to 0} T(z) = \lim_{z\to 0} \frac{1}{z} = \infty$$

since $\lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{1}{z} = 0 \rightarrow (4)$

~~since~~ and $\lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{1}{z} = 0$

since $\lim_{z\to\infty} T\left(\frac{1}{z}\right) = \lim_{z\to\infty} z = \infty \rightarrow (5)$

In order to make T is continuous on the extended plane and we write.
$T(0) = \infty$, $T(\infty) = 0$ & $T(z) = \frac{1}{z}$ for the remaining values of $z$.

More precisely, the first limit of (4) & (5) we have $\lim_{z\to z_0} T(z) = T(z_0)$ which is clearly prove when $z_0 \neq 0$.

And when $z_0 \neq \infty$, is also prove for those two values of $z_0$ the fact that T is continuous everywhere in the extended plane.

**Linear fractional transformation:-**

The transformation of the form

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

where a, b, c, d are complex constants is known as linear fractional transformation (or) ~~mobius~~ ~~mobius~~ mouious transformation.

since,
$$w(cz+d) = az+b \rightarrow (1)$$
$$wcz + wd = az+b$$
$$wcz - az = b - wd$$
$$(wc-a)z = b - wd \rightarrow (2)$$

$z = \frac{b-wd}{wc-a}$

eqn (1) & (2) to choose the ~~mobius~~ movious transformation is linear both in $w$ & $z$.

Hence it is also known as ~~the~~ bilinear transformation.

→ Bilinear transformation satisfies

S.T a linear transformation is a combination of Both elementary transformation.

**proof**

5M

W.K.T $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

$$= \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)} = \frac{a}{c}\left[\frac{z+\frac{d}{c}-\frac{d}{c}+\frac{b}{a}}{z+\frac{d}{c}}\right]$$

$$= \frac{\frac{a}{c}\left[\left(z+\frac{d}{c}\right)+\left(\frac{b}{a}-\frac{d}{c}\right)\right]}{z+\frac{d}{c}}$$

$$= \frac{a}{c} + \frac{\frac{a}{c}\left[\frac{b}{a}-\frac{d}{c}\right]}{z+\frac{d}{c}}$$

$$= \frac{a}{c} - \frac{\frac{\cancel{a}}{c}\left[\frac{ad-bc}{\cancel{a}c}\right]}{z+\frac{d}{c}}$$

$$= \frac{a}{c}\left[1+\frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}\right]$$

$$= \frac{a}{c}\left[1+\frac{\frac{bc-da}{ac}}{z+\frac{d}{c}}\right]$$

$$= \frac{a}{c} + \frac{\frac{a}{c}\left[\frac{bc-da}{ac}\right]}{z+\frac{d}{c}}$$

$$= \frac{a}{c} + \frac{1}{c^2}\left[\frac{bc-ad}{z+\frac{d}{c}}\right]$$

Let $T_1 = z+\frac{d}{c}$ it is translation.

$T_2 = \frac{1}{T_1}$ it is inversion.

$T_3 = \left[\frac{bc-ad}{c^2}\right]T_2$ it is magnification.

$T_4 = \frac{a}{c} + T_3$ it is translation.

Hence the bilinear transformation is a combination of translation inversion and magnification.

Note :-

If we write $T(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

then $T(\infty) = \infty$ if $c=0$

$= \frac{a}{c}$ if $c \neq 0$.

Result: If $w = T(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is bilinear transformation then $z = T^{-1}(w)$ is also bilinear transformation.

Proof :

Let $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is a bilinear transformation.

$w(cz+d) = az+b$

$wcz + wd = az+b$.

$wcz - az = b - wd$

$(wc-a)z = b - wd$

$z = \frac{b-wd}{wc-a}$

$T^{-1}(w) = z = \frac{-dw+b}{cw-a}$ is a bilinear transformation

if $\begin{vmatrix} -d & b \\ c & -a \end{vmatrix} \neq 0$ is $ad-bc$

Hence $w$ is a bilinear transformation and hence $z = T^{-1}(w)$ is also a ~~tra~~ bilinear transformation.

P.T combination of bilinear tranformation is also bilinear transformation (or) composition of two ~~two~~ linear transformation is also a linear transformation.

Proof :-

~~Since~~ Let $T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, $a_1 d_1 - b_1 c_1 \neq 0$

$T_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$, $a_2 d_2 - b_2 c_2 \neq 0$

Be two linear Transformation,

$$(T_1 \circ T_2)(z) = T_1(T_2(z))$$

$$= T_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right)$$

$$= \frac{a_1\left[\dfrac{a_2 z + b_2}{c_2 z + d_2}\right] + b_1}{c_1\left[\dfrac{a_2 z + b_2}{c_2 z + d_2}\right] + d_1}$$

$$= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)}.$$

$$= \frac{a_1 a_2 z + a_1 b_2 + b_1 c_2 z + b_1 d_2}{c_1 a_2 z + c_1 b_2 + d_1 c_2 z + d_1 d_2}$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2) z + c_1 b_2 + d_1 d_2}.$$

$$= \frac{A z + B}{C z + D}.$$

where, $A = a_1 a_2 + b_1 c_2$

$B = a_1 b_2 + b_1 d_2$

$C = c_1 a_2 + d_1 c_2$

$D = c_1 b_2 + d_1 d_2.$

consider,

$$AD - BC = (a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (a_1 b_2 + b_1 d_2)(c_1 a_2 + d_1 c_2).$$

$$= (a_1 a_2 c_1 b_2 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + b_1 c_1 d_1 d_2) - (a_1 b_2 c_1 a_2 + a_1 b_1 d_1 c_2 +$$

$$b_1 d_2 c_1 a_2 + \ldots$$

$= a_1 d_1 (a_2 d_2 - b_2 c_2) + b_1 c_1 (b_2 c_2 - a_2 d_2)$

$= a_1 d_1 (a_2 d_2 - b_2 c_2) - b_1 c_1 (a_2 d_2 - b_2 c_2)$

$= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0.$

$\therefore a_1 d_1 - b_1 c_1 \neq 0$ & $a_2 d_2 - b_2 c_2 \neq 0.$

Hence $T_1 \circ T_2$ is also bilinear.

Transformation critical point :-

The point $z = \frac{-b}{a}$ and $z = \frac{-d}{c}$ which corresponds to $w = 0$ and $w = \infty$ are critical point of bilinear transformation.

Illy the point $w = \frac{b}{d}$ and $w = \frac{a}{c}$ which corresponds to $z = 0$ & $z = \infty$ are also critical points.

Theorem:

P/T a bilinear transformation takes circle (or) straight lines to circles (or) straight line and inverse points to inverse point.

proof:

W.R.T Example of circle having inverse point p and q can be written as

$$\left| \frac{z - p}{z - q} \right| = k, \quad k \neq 1 \rightarrow ①$$

Eqn of straight line

$y = mx + c$

and

Eqn of circle

$(x-a)^2 + (y-b)^2 = r^2$

If $k = 1$ then eqn ① represent a straight line.

Let $w = \frac{az + b}{cz + d}$, $ad - bc \neq 0$

$wcz + wd = az + b$

$wcz - az + wd - b = 0.$

$$(wc - a)z = b - dw$$

$$z = \frac{b-dw}{wc-a} \longrightarrow (2)$$

sub (2) in (1)

$$(1) \Rightarrow \frac{\left|\frac{b-dw}{wc-a} - p\right|}{\left|\frac{b-dw}{wc-a} - q\right|} = k.$$

$$\frac{|b-dw-p(wc-a)|}{|b-dw-q(wc-a)|} = k$$

$$\frac{|w(-d-pc) + pa+b|}{|w(-d-qc) + qa+b|} = k$$

$$\Rightarrow \frac{|d+pc|\left|w-\left(\frac{pa+b}{d+pc}\right)\right|}{|d+qc|\left|w-\frac{qa+b}{d+qc}\right|} = k$$

$$\frac{\left|w-\left(\frac{pa+b}{d+pc}\right)\right|}{\left|w-\left(\frac{qa+b}{d+qc}\right)\right|} = k\frac{|d+qc|}{|d+pc|}.$$

$$\left|\frac{w-p'}{w-q'}\right| = k' \longrightarrow (3)$$

where $p' = \frac{b+pa}{d+pc}$, $q' = \frac{b+qa}{d+pc.}$, $k' = k\frac{|d+qc|}{|d+pc|}$

$\therefore$ eqn (3) represent a circle in $\omega$-plane where $p'$ & $q'$ are inverse point.

condition 1:-

If $k=1$ and $k'=1$ the bilinear transformation transforms straight line to straight line and symmetric point to symmetric point.

condition 2:-

If $k=1$ & $k' \neq 1$; the straight line is transformed to circle and symmetric point are mapped onto inverse point.

Condition 3:-

If $k \neq 1$ & $k'=1$, The circle mapped onto a straight line & inverse point are mapped onto symmetric point.

condition 4:-

If $k \neq 1$ and $k' \neq 1$, the bilinear transformation transforms circle to circle and inverse point to inverse point.

Find the bilinear transformation which maps $z_1=-1$, $z_2=0$, $z_3=1$ onto the points $w_1=-i$, $w_2=1$, $w_3=i$

Soln

w.k.T $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

$z_1=-1$, $w_1=-i$.

$$-i = \frac{a(-1)+b}{c(-1)+d}$$

$$-i(-c+d) = -a+b$$

$$ci - di = -a+b \rightarrow ①$$

$z_2=0$, $w_2=1$

$$1 = \frac{b}{d}$$

$$d = b \rightarrow ②$$

$z_3 = 1,\ w_3 = i$

$$i = \frac{a(1)+b}{c(1)+d}$$

$$i = \frac{a+b}{c+d}$$

$ci + di = a + b \longrightarrow (5)$

$ci - di = -a + b \longrightarrow (1)$

$\cancel{2}ci = \cancel{2}b$

$ci = b$

$c = \frac{b}{i} \times \frac{-i}{-i}$

$c = -bi$

$(1) \Rightarrow ci - di = -a + b$

$(-bi)i - di = -a + b$

$b - di = -a + b$

$b - bi = -a + b$

$-bi = -a + b - b$

$-bi = -a$

$bi = a$

$\boxed{a = bi}$

$$\therefore w = \frac{biz + b}{-biz + b} = \frac{iz+1}{-iz+1} = \frac{i(z-i)}{i(-z-i)} = \frac{z-i}{-z-i}$$

$$\therefore w = \frac{z-i}{-(+z+i)}$$

$$= \frac{-z+i}{z+i}$$

$$w = \frac{i-z}{z+i}$$

Find the bilinear transformation which maps the points $z_1=1$, $z_2=0$, $z_3=-1$ onto the points $w_1=i$, $w_2=\infty$, $w_3=1$.

W.K.T $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$z_1=1$, $z_2=0$, $z_3=-1$ and $w_1=i$, $w_2=\infty$, $w_3=1$

$$\frac{(w-i)(\infty-1)}{(i-\infty)(1-w)} = \frac{(z-1)(0+1)}{(1-0)(-1-z)}$$

$$\frac{(w-i)\infty(1-\frac{1}{\infty})}{\infty(\frac{i}{\infty}-1)(1-w)} = \frac{(z-1)}{(-1-z)}$$

$$\frac{(w-i)}{(1-w)(-1)} = \frac{z-1}{-(1+z)}$$

$$(w-i)(1+z) = (z-1)(1-w)$$

$$w+wz-i-iz = z-zw-1+w.$$

$$wz+zw = z-1+i+iz$$

$$2wz = z(1+i)-1+i$$

$$w = \frac{z(1+i)-(1-i)}{2z}.$$

An implicit form:

The equation

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \rightarrow (1)$$

defines a linear fractional transformation that maps distinct point transformation points $z_1$, $z_2$ and $z_3$ in the finite z-plane onto distinct points $w_1$, $w_2$ and $w_3$ respectively in the finite w-plane.

Find the linear transformation maps $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ to $\omega_1 = -i$, $\omega_2 = 1$, $\omega_3 = i$ by using implicit form:-

w.k.T. $$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

$$\frac{(\omega+i)(1-i)}{(\omega-i)(1+i)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}$$

$$\frac{(\omega+i)(1-i)}{(\omega-i)(1+i)} = \frac{(z+1)(-1)}{z-1}$$

$$\frac{\omega+i-\omega i+1}{\omega-i+\omega i+1} = \frac{(z+1)(-1)}{z-1} = \frac{z+1}{1-z}.$$

using componendo and dividendo.

$$\frac{\omega(1-i)+(1+i)+\omega(1+i)+(1-i)}{\omega(1-i)+(1+i)-[\omega(1+i)+(1-i)]} = \frac{(z+1)+(1-z)}{(z+1)-(1-z)},$$

$$\frac{\omega-\omega i+1+i+\omega+\omega i+1-i}{\omega-\omega i+1+i-\omega-\omega i-1+i} = \frac{\cancel{z}+1+1-\cancel{z}}{z+\cancel{1}-\cancel{1}+z}$$

$$\frac{2+2\omega}{-2\omega i+2i} = \frac{2}{2z}$$

$$\frac{1+\omega}{-(\omega-1)i} = \frac{1}{z}$$

$$z = \frac{-\omega i+i}{1+\omega}$$

$$(1+\omega)z = -\omega i+i$$

$$z+z\omega = -\omega i+i$$

$$z\omega+\omega i = i-z$$

$$\omega(z+i) = i-z$$

$$\omega = \frac{i-z}{z+i},$$

cross ratio :-

A cross ratio of four points $z_1, z_2, z_3, z_4$ is defined as $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ and it is denoted by $(z_1, z_2, z_3, z_4)$.

Theorem:-

A bi linear transformation preserves cross ratio (or) cross ratio is invariant under bi linear transformation

proof:-

Let $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ be a bi linear transformation

consider, $w_i - w_j = \frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d}$

$$= \frac{(az_i+b)(cz_j+d)-(az_j+b)(cz_i+d)}{(cz_i+d)(cz_j+d)}$$

$$= \frac{ac\cancel{z_iz_j} + az_id + bcz_j + \cancel{bd} - a\cancel{z_jcz_i} - adz_j - bcz_i - \cancel{bd}}{(cz_i+d)(cz_j+d)}$$

$$= \frac{ad(z_i-z_j) - bc(z_i-z_j)}{(cz_i+d)(cz_j+d)}$$

$$= \frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}.$$

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_4-w_1)(w_2-w_3)} = \frac{\frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}}{\frac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)} \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}}$$

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_4-w_1)(w_2-w_3)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_4-z_1)(z_2-z_3)}.$$

Mapping of the upper half plane:

Statement :-

The bilinear transformation which maps $\operatorname{Im} z$ [upper half plane] onto the unit is map $|w| \le 1$ is given by $w = e^{i\lambda}\left[\frac{z-\alpha}{z-\bar{\alpha}}\right]$ provided $\operatorname{Im} z > 0$.

proof :-

Any bilinear transformation is of the form $w = \frac{az+b}{cz+d} \rightarrow (1)$ $ad-bc \neq 0$.

W.K.T bilinear transformation transforms circle or straight line to circle or straight line and inverse point to inverse point.

Here the real axis $\operatorname{Im} z = 0$ is transformed to unit circle $|w|=1$.

As $0$ and $\infty$ are inverse point w.r.to $|w|=1$, their pre images $z = \frac{-b}{a}$ & $z = \frac{-d}{c}$ are symmetric points.

Let $\alpha = \frac{-b}{a}$ & $\bar{\alpha} = \frac{-d}{c}$.

Hence (1) takes the form.

$$\therefore w = \frac{az+b}{cz+d} = \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)}$$

$$w = \frac{a}{c}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \longrightarrow (2)$$

As $z=0$ is preimage of same point on $|w|=1$

we have $1 = |w| = \left|\frac{a}{c}\right| \frac{|0-\alpha|}{|0-\bar{\alpha}|}$

$$1 = \left|\frac{a}{c}\right| \qquad |\alpha| = |\bar{\alpha}|$$

$$\frac{a}{c} = e^{i\lambda}$$

where $\lambda$ is real, Thus the transformation takes the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$

Now $z=\infty$ is preimage of $w=0$, since $w=0$ being centre of the circle $|w|=1$ the preimage should be interior point of the domain. i.e) ~~...~~. $\operatorname{Im} \alpha > 0$

which takes $\operatorname{Im} z \geq 0$.

Thus bilinear transformation on the unit disk given by $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ provided

$\operatorname{Im} z > 0$.

Verify:-

To verify $w = e^{i\lambda}$ is the transformation takes

$\operatorname{Im} z \geq 0$ onto $|w| \leq 1$.

consider.

$$w\bar{w} - 1 = \frac{e^{i\lambda}(z-\alpha)}{(z-\bar{\alpha})} \cdot \frac{e^{-i\lambda}\,\overline{(z-\alpha)}}{\overline{(z-\bar{\alpha})}} - 1$$

$$= \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\bar{\alpha})(\bar{z}-\alpha)} - 1$$

$$= \frac{z\bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha}}{|z-\bar{\alpha}|^2} - 1$$

$$= \frac{(z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha}) - (z - \bar{\alpha})(\bar{z} - \alpha)}{|z-\bar{\alpha}|^2}$$

$$= \frac{\cancel{z\bar{z}} - \alpha\bar{z} - \bar{\alpha}z + \cancel{\alpha\bar{\alpha}} - \cancel{z\bar{z}} + \bar{\alpha}\bar{z} + \alpha z - \cancel{\alpha\bar{\alpha}}}{|z-\bar{\alpha}|^2}$$

$$= \frac{z(\alpha - \bar{\alpha}) + \bar{z}(\bar{\alpha} - \alpha)}{|z-\bar{\alpha}|^2} = \frac{(\alpha-\bar{\alpha})(z-\bar{z})}{|z-\bar{\alpha}|^2}$$

$$= \frac{(2i\, \mathrm{Im}(z)) \cdot 2i\, \mathrm{Im}\,\alpha}{|z-\bar{\alpha}|^2} = \frac{-4(\mathrm{Im}\, z)(\mathrm{Im}\,\alpha)}{|z-\bar{\alpha}|^2}$$

$\mathrm{Im}\, z = 0$ the real axis is transformed

to $w\bar{w} = 1 = 0$ ie $|w| = 1$

$w\bar{w} - 1 \leq 0$

$\Rightarrow \frac{-4(\mathrm{Im}\, z)(\mathrm{Im}\,\alpha)}{|z-\bar{\alpha}|} < 0$

$\Rightarrow \mathrm{Im}\, z > 0$

ie The upper half plane is mapped

onto $|w| \leq 1$

Hence $w = \frac{e^{i\lambda}(z-\alpha)}{(z-\bar{\alpha})}$

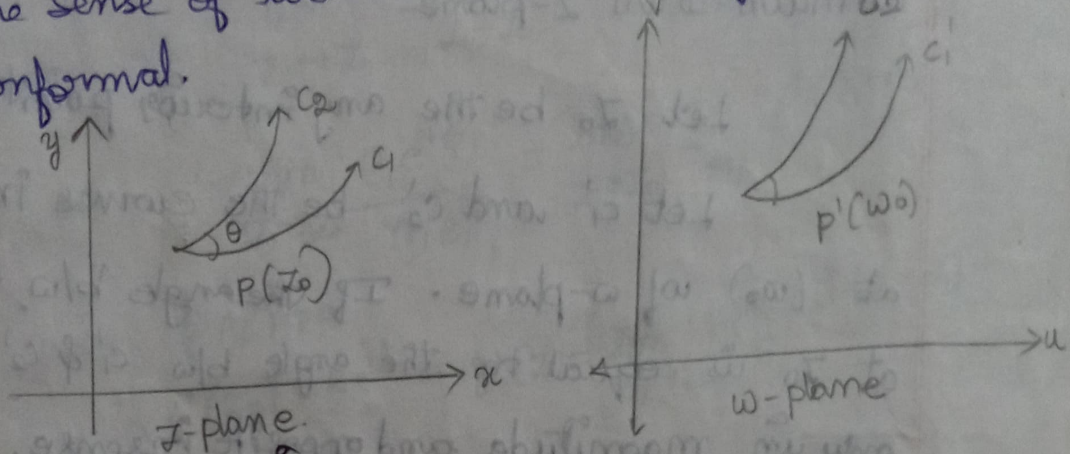
$\mathrm{Im}\,\alpha > 0$ is the transformation on which maps $\mathrm{Im}\, z \geq 0$ onto the unit is

$|w| < 1$.

Conformal mapping:-

Eg $w = e^{z}$.

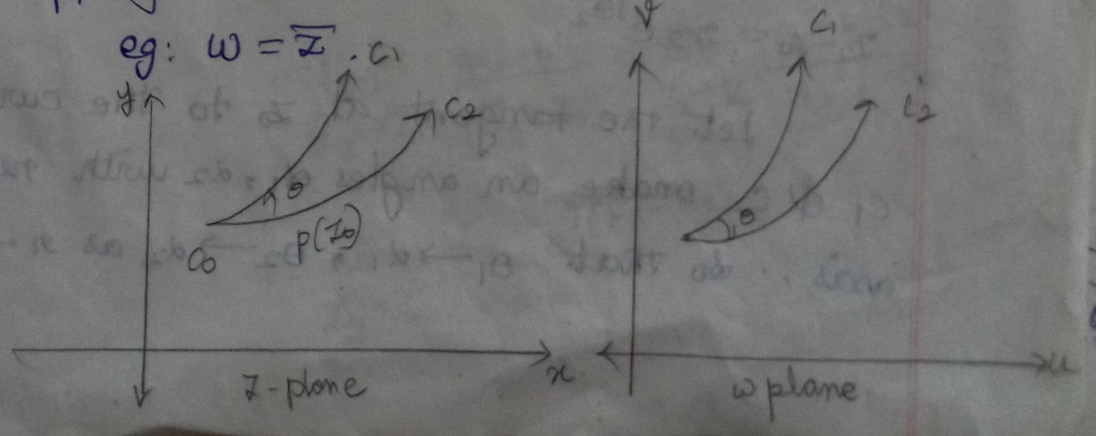
Suppose the transformation $u = v(x,y)$ and $v = u(x,y)$ maps the two curves $c_1, c_2$ intersecting at the point $p(z_0)$ of $z$-plane onto two curves $c_1', c_2'$ intersecting $p'(w_0)$ of $w$-plane. If the angles b/w $c_1$ and $c_2$ at $z_0$ is equal to the angle b/w $c_1'$ and $c_2'$ at $w_0$ both in magnitude & the sense of location the mapping is said to be conformal.




Isogonal mapping:-

Suppose that the transformation $U = u(x,y)$ and $V = v(x,y)$ maps the two curves $c_1, c_2$ intersecting at the point $p(z_0)$ of $z$-plane onto the curves $c_1', c_2'$ intersecting at $p'(z_0)$ $p'(w_0)$ of $w$-plane. If the angle b/w $c_1$ & $c_2$ at $z_0$ is equal to the angle b/w $c_1'$ & $c_2'$ at $w_0$ only in magnitude & opposite in sense, the mapping is said to be Isogonal mapping.

eg: $w = \bar{z}$



## Sufficient Condition for conformal mapping:

W

**Theorem:**

10m If $w = f(z)$ is analytic and $f'(z) \neq 0$ in a domain $D$ then the mapping $w = f(z)$ is conformal in $D$.

**Proof:**

Let $w = f(z)$ be an analytic function in a domain $D$ in $z$-plane.

Let $z_0$ be the any interior point of $D$.

Let $c_1'$ and $c_2'$ be the curves intersecting at $(w_0)$ of $w$-plane. If the angle b/w $c_1$ & $c_2$ & the angle b/w $c_1'$ & $c_2'$ at $w_0$ at $z_0$ is equal to the angle b/w $c_1'$ & $c_2'$ at $w_0$ at $z_0$ is equal only in magnitude and opposite in sense.

The corresponding two curves $c_1$ & $c_2$ intersecting at $z_0$ in $z$-plane

Let $w_1$ & $w_2$ be the point on the $c_1'$ & $c_2'$ respect to the corresponding points on $z_1, z_2$ on $c_1$ & $c_2$ respectively $\ni$ the distance b/w $z_1$ & $z_0$ is equal to the distance b/w $z_2$ & $z_0$ (say $r$).

$\therefore$ we can write $z_1 - z_0 = r_1 e^{i\theta_1}$ &

$z_2 - z_0 = r_2 e^{i\theta_2}$.

Let the tangent at $z_0$ to the curves $c_1$ & $c_2$ make an angles $\alpha_1, \alpha_2$ with real axis, so that $\theta_1 \to \alpha_1$, $\theta_2 \to \alpha_2$ as $r \to 0$.

Also let the tangent at $w_0$ to the curve $c_1'$ & $c_2'$ make angle $\beta_1$ & $\beta_2$ with real axis,

Let $w_1 - w_0 = \rho_1 e^{i\phi_1}$ and

$w_2 - w_0 = \rho_2 e^{i\phi_2}$ where $\phi_1 \to \beta_1$ as $\rho_1 \to 0$

and $\phi_2 \to \beta_2$ as $\rho_2 \to 0$.

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0}$$

$$= \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} \Rightarrow \lim_{z_1 \to z_0} \frac{w_1 - w_0}{r_1 e^{i\theta_1}}$$

$$= \lim_{z_1 \to z_0} \frac{\rho_1 e^{i\phi_1}}{r_1 e^{i\theta_1}} = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\phi_1 - \theta_1)}$$

given, $f'(z) \neq 0$

$\therefore f'(z_0) = Re^{i\lambda} \Rightarrow f'(z) = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\beta_1 - \alpha_1)}$.

Then $Re^{i\lambda} = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\beta_1 - \alpha_1)}$

$\Rightarrow R = \frac{\rho_1}{r_1}$ as $z_1 \to z_0$ and

$\lambda = \lim_{z_1 \to z_0} \beta_1 - \alpha_1 \Rightarrow R = \frac{\rho_1}{r_1}$

$\lambda = \beta_1 - \alpha_1 \longrightarrow (1)$

IIIy $f'(z_0) = \lim_{z_2 \to z_0} \frac{f(z_2) - f(z_0)}{z_2 - z_0}$

$$= \lim_{z_2 \to z_0} \frac{w_2 - w_0}{z_2 - z_0} = \lim_{z_2 \to z_0} \frac{\rho_2}{r_2} \frac{e^{i\phi_2}}{e^{i\theta_2}}$$

$$= \lim_{z_2 \to z_0} \frac{\rho_2}{r_2} e^{i(\beta_2 - \alpha_2)}.$$

And we have, $\lambda = \lim_{z_2 \to z_0} \beta_2 - \alpha_2$

$$\lambda = \beta_2 - \alpha_2 \rightarrow (2)$$

From (1) & (2)

$$\beta_1 - \alpha_1 = \beta_2 - \alpha_2$$

$$\beta_1 - \beta_2 = \alpha_1 - \alpha_2$$

i.e) The angle b/w $c_1'$ & $c_2'$ at $w_0$ is equal in magnitude as well as in sign to the angle b/w the curves $c_1$ & $c_2$ at $z_0$.

Thus the mapping $w = f(z)$ is conformal.

Hence proved.